Sec 5.3 Homog. eqs w/ const coeffs (part 1)

Previously:

(ex) $y'' - y' - 6y = 0$

"try" solution $y = e^{rx}$, $\Rightarrow y' = re^{rx}$, $y'' = r^2 e^{rx}$

plug in to ODE $\rightarrow r^2 e^{rx} - re^{rx} + 6e^{rx} = 0$

$$e^{rx}(\underline{r^2 - r + 6}) = 0$$

"characteristic eqn"

$$r^2 - r - 6 = 0$$
$$(r-3)(r+2) = 0$$

roots are $r_1 = -2$, $r_2 = 3$ (so $r_1 \neq r_2$.)

roots occur once each...

$\Rightarrow$ "roots $r = -2$ and $r = 3$ have multiplicity 1"

gen solution $y = c_1 \boxed{e^{-2x}} + c_2 \boxed{e^{3x}}$

lin indep

(ex) $y'' - 6y' + 9y = 0$

ch. eq. $r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

roots are $r_1 = 3$, $r_2 = 3$ $(r_1 = r_2 = 3)$

"root $r = 3$ has multiplicity 2"

$1 = 2 - 1$

gen solution: $y = (c_1 + c_2 x^{1}) e^{3x}$

Sec 5.3 Homog. eqs w/ const coeffs (part 1)

- Pattern/methods works for higher order homog eqs if coeffs of y, y', y'', etc. are const.

Ex $1y^{(5)} - 8y^{(4)} + 16y^{(3)} = 0$ ← homog

const coeffs + homog ⟹ find roots/factor char. eq.

char. eq.: $r^5 - 8r^4 + 16r^3 = 0$

$$r^3(r^2 - 8r + 16) = 0$$

$$r^3(r-4)(r-4) = 0$$

roots are $r = \underbrace{0, 0, 0}_{\text{multiplicity } 3}, \underbrace{4, 4}_{\text{mult. } 2}$

multiplicity = "# of occurences"

5^{th} order eqn ⟹ need 5 lin. indep "basis funcs"

gen sol $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 + c_5 y_5$

Since $r = 0, 0, 0, 4, 4$, our gen sol looks like

$$y = [\text{"}r=0 \text{ piece"}] + [\text{"}r=4 \text{ piece"}]$$

$$= [(c_1 + c_2 x + c_3 x^2)e^{0x}] + [(c_4 + c_5 x^1)e^{4x}]$$

b/c $r=0$ has mult. 3 $\qquad$ $r=4$ has mult 2

$(e^{0x} = 1)$ $y = c_1 \underbrace{\boxed{1}}_{y_1} + c_2 \underbrace{\boxed{x}}_{y_2} + c_3 \underbrace{\boxed{x^2}}_{y_3} + c_4 \underbrace{\boxed{e^{4x}}}_{y_4} + c_5 \underbrace{\boxed{xe^{4x}}}_{y_5}$

Can "work backwards":

Ex: Write a homog. linear ODE with gen solution

$$y = \underbrace{c_1}_{\substack{r=0 \\ \text{mult. } 1}} + \underbrace{c_2 e^{-x} + c_3 x e^{-x}}_{\substack{r=-1 \\ \text{mult. } 2}} + \underbrace{c_4 e^{5x}}_{\substack{r=5 \\ \text{mult. } 1}}$$

$\Longrightarrow$ char. eqn should be $(r-0)(r+1)^2(r-5)=0$

$$\longrightarrow r(r^2+2r+1)(r-5)=0$$

$$r(r^3+2r^2+r-5r^2-10r-5)=0$$

$$r^4-3r^3-9r^2-5r=0$$

So original ODE should be

$$y^{(4)}-3y^{(3)}-9y''-5y'=0$$

$\left(r^k \longleftrightarrow y^{(k)}, \quad \text{const} \longleftrightarrow (\text{const})y \right)$

<u>Thm</u> General sols of ODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

(homog, linear, const coeffs)

are $\quad y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

$\{y_1, y_2, \ldots, y_n\}$ must be linearly indep.

- Find $y_1, \ldots, y_n$ via <u>char. eq.</u>:

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Solving (cubics +) by "inspection/division"

Sometimes find "pattern"

ch. eq. $\underset{\times 5}{1r^3 + 5r^2} - \underset{\times 5}{100r - 500} = 0$

so maybe group ... $r^2(r+5) - 100(r+5) = 0$

$(r+5)(r^2 - 100) = 0$

$\Rightarrow \quad r = -5, \; r = 10, -10.$

Educated guess + division:

Ex: $3y^{(3)} + 8y'' - 33y' + 10y = 0$

ch. eq: $3r^3 + 8r^2 - 33r + 10 = 0$

Hope for $(r-k)\underbrace{(ar^2 + br + c)}_{\text{can use quad formula.}} = 0$

need this root

Recall Theorem: $a_n r^n + \cdots + a_1 r + a_0 = 0$

any rational root $r = \frac{p}{q}$ (p, q coprime)

must have $p = \pm(\text{divisor of } a_0)$

$q = \pm(\text{divisor of } a_n)$

(this gives a candidate list)

for $3r^3 + 8r^2 - 33r + 10 = 0$

rational root candidates $\frac{p}{q} \begin{array}{l} \rightarrow \in \{\pm 1, \pm 2, \pm 5, \pm 10\} \\ \hline \rightarrow \in \{\pm 1, \pm 3\} \end{array}$

* always start with $r = \frac{p}{1}$ ($q=1 \Rightarrow$ integer) (easier)

plug in to ch. eq.

$r = 1 \longrightarrow 3 + 8 - 33 + 10 = -12$ ✗

$r = -1 \longrightarrow -3 + 8 + 33 + 10 = 48$ ✗

$r = 2 \longrightarrow 24 + 32 - 66 + 10 = 0$ ✓

so $r=2$ is a root

$\Rightarrow 3r^3 + 8r^2 - 33r + 10 = 0$

$= (r-2)(\underline{ar^2 + br + c}) = 0$

need to find by polynomial long division

$$
\begin{array}{r|l}
 & 3r^2 + 14r - 5 \\ \hline
r-2 & 3r^3 + 8r^2 - 33r + 10 \\
 & \underline{3r^3 - 6r^2} \\
 & \quad 14r^2 - 33r \\
 & \quad \underline{14r^2 - 28r} \\
 & \qquad -5r + 10 \\
 & \qquad \underline{-5r + 10} \\
 & \qquad\qquad 0
\end{array}
$$

ch. eq. : $(r-2)(3r^2 + 14r - 5) = 0$

$(r-2)(r+5)(3r-1) = 0$

$r = 2 \quad r = -5 \quad r = 1/3$